Maximally and non-maximally fast escaping points of transcendental entire functions

Journal Item

Maximally and non-maximally fast escaping points of transcendental entire functions

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(*Received* 5 *December* 2014)

Abstract

We partition the fast escaping set of a transcendental entire function into two subsets, the maximally fast escaping set and the non-maximally fast escaping set. These sets are shown to have strong dynamical properties. We show that the intersection of the Julia set with the non-maximally fast escaping set is never empty. The proof uses a new covering result for annuli, which is of wider interest.

It was shown by Rippon and Stallard that the fast escaping set has no bounded components. In contrast, by studying a function considered by Hardy, we give an example of a transcendental entire function for which the maximally and non-maximally fast escaping sets each have uncountably many singleton components.

1. *Introduction*

Suppose that $f : \mathbb{C} \to \mathbb{C}$ is a transcendental entire function. The *Fatou set* $F(f)$ is defined as the set $z \in \mathbb{C}$ such that $\{f^n\}_{n\in\mathbb{N}}$ is a normal family in a neighbourhood of z. The *Julia set* $J(f)$ is the complement in $\mathbb{C}$ of $F(f)$. An introduction to the properties of these sets was given in [3].

For a general transcendental entire function the *escaping set*

$$I(f) = \{z : f^n(z) \to \infty \text{ as } n \to \infty\}$$

was studied first in [7]. This paper concerns a subset of $I(f)$, the *fast escaping set* $A(f)$. This was introduced in [5], and can be defined [21] by

$$A(f) = \{z : \text{there exists } \ell \in \mathbb{N} \text{ such that } |f^{n+\ell}(z)| \geq M^n(R, f), \text{ for } n \in \mathbb{N}\}. \tag{1.1}$$

Here the *maximum modulus function* is defined by $M(r, f) = \max_{|z|=r} |f(z)|$, for $r \geq 0$. We write $M^n(r, f)$ to denote repeated iteration of $M(r, f)$ with respect to the variable r. In (1.1), $R > 0$ is such that $M^n(R, f) \to \infty$ as $n \to \infty$.

A major open question in transcendental dynamics is the conjecture of Eremenko [7] that, for every transcendental entire function f, $I(f)$ has no bounded components.

2010 *Mathematics Subject Classification.* Primary 37F10; Secondary 30D05.

The author was supported by Engineering and Physical Sciences Research Council grant EP/J022160/1.

A significant result regarding this conjecture was given by Rippon and Stallard, who showed [21, Theorem 1.1] that $A(f)$ has no bounded components. In view of this, the fast escaping set has been widely studied in recent years; see, for example, the papers [12, 16–21, 23] and [25].

We introduce a partition of $A(f)$, and show that the two sets in this partition share many properties with $A(f)$. On the other hand, we show that there is a transcendental entire function such that the components of these sets have unexpected boundedness properties.

First we define

$$A'(f) = \{z \in A(f) : \exists N \in \mathbb{N} \text{ s.t. } |f^n(z)| = M(|f^{n-1}(z)|, f), \text{ for } n \geq N\}, \tag{1.2}$$

and we let

$$A''(f) = A(f) \backslash A'(f).$$

We describe $A'(f)$ as the *maximally fast escaping set*, and $A''(f)$ as the *non-maximally fast escaping set.* A set S is *completely invariant* if $z \in S$ implies that $f(z) \in S$ and $f^{-1}(\{z\}) \subset S$. It is clear from the definitions that both $A'(f)$ and $A''(f)$ are completely invariant.

Our first result shows that, in some sense, $A'(f)$ is at most a small set.

THEOREM 1. *If f is a transcendental entire function, then $A'(f)$ is contained in a countable union of curves each of which is analytic except possibly at its endpoints.*

In Example 1 we give a transcendental entire function such that $A'(f)$ has a single unbounded component, consisting of a countable union of analytic curves. It follows from Theorem 1 that this example is, in this sense, maximal.

It was shown in [5] that it follows from the construction in [7] that $A(f) \neq \emptyset$. In [17, Remark 2] it was further shown that $A(f) \cap J(f) \neq \emptyset$. In Example 2 we give a transcendental entire function such that $A'(f) = \emptyset$. On the other hand, using a new annuli covering result and a recent result regarding the properties of the boundary of a multiply connected Fatou component, we show that $A''(f) \cap J(f) \neq \emptyset$; roughly speaking, this means that there are always points in the Julia set for which the rate of escape is "fast" but not "maximally fast".

THEOREM 2. *If f is a transcendental entire function, then $A''(f) \cap J(f) \neq \emptyset$.*

It is known [21, Theorem 5.1(c)] that $A(f)$ is dense in $J(f)$, and [21, Theorem 1.2] that if U is a Fatou component that meets $A(f)$, then U is contained in $A(f)$. In the following theorem we strengthen these facts, and show that the sets $A'(f)$ and $A''(f)$ have strong dynamical properties relating to the Fatou and Julia sets. Note [21, Theorem 4.4] that all multiply connected Fatou components are in $A(f)$, and [4, 24] there exist transcendental entire functions with simply connected Fatou components contained in $A(f)$.

THEOREM 3. *Suppose that f is a transcendental entire function. Then the following all hold.*

(a) If U is a simply connected Fatou component of f such that $U \cap A(f) \neq \emptyset$, then $U \subset A''(f)$.

(b) If U is a multiply connected Fatou component of f, then $U \cap A''(f) \neq \emptyset$.

(c) $A''(f)$ is dense in $J(f)$ and $J(f) \subset \partial A''(f)$.
(d) If $A'(f) \cap J(f) \neq \emptyset$, then $A'(f)$ is dense in $J(f)$ and $J(f) \subset \partial A'(f)$.
(e) If $A'(f) \cap F(f) = \emptyset$, then $J(f) = \partial A''(f)$.
(f) If $A'(f) \cap J(f) \neq \emptyset$ and $A'(f) \cap F(f) = \emptyset$, then $J(f) = \partial A'(f)$.

Clearly it follows from part (a) that if f has no multiply connected Fatou component, then $A'(f) \cap F(f) = \emptyset$. The additional hypothesis that $A'(f) \cap F(f) = \emptyset$ in parts (e) and (f) of Theorem 3 is required; in Example 3 we give a transcendental entire function, f, such that

$$A'(f) \cap F(f) \neq \emptyset, \; J(f) \neq \partial A''(f) \text{ and } J(f) \neq \partial A'(f).$$

In Example 4 we give a transcendental entire function, f, which has a multiply connected Fatou component and which satisfies $A'(f) = \emptyset$.

We recall the property, mentioned earlier, that $A(f)$ has no bounded components. By studying a function considered by Hardy we show that this property does not hold for $A'(f)$ or $A''(f)$.

Theorem 4. *There is a transcendental entire function f such that*
(a) $A'(f)$ is uncountable and totally disconnected;
(b) $A''(f)$ has uncountably many singleton components and at least one unbounded component.

On the other hand, in Example 6 we give a transcendental entire function such that $A(f)$ has an unbounded component which is contained in $A'(f)$.

The structure of this paper is as follows. To prove Theorem 2 we require a new covering result for annuli, which is of independent interest. This is given in Section 2. In Section 3 we prove Theorem 1 and Theorem 2. In Section 4 we prove Theorem 3. Finally, in Section 5 we give the examples; the proof of Theorem 4 is included in Example 5.

2. *A new covering result*

In this section we give a new covering result for annuli, which is used to construct a point in $A''(f) \cap J(f)$ in the case that f is a transcendental entire function with no multiply connected Fatou component. This result is similar to [15, Theorem 2.2], generalised to the whole family of iterates of a transcendental entire function. Here the *minimum modulus function* is defined, for $r \geq 0$, by $m(r, f) = \min_{|z|=r} |f(z)|$. We also use the notation for an annulus

$$A(r_1, r_2) = \{z : r_1 < |z| < r_2\}, \quad \text{for } 0 < r_1 < r_2,$$

and a disc

$$B(\zeta, \; r) = \{z : |z - \zeta| < r\}, \quad \text{for } 0 < r, \; \zeta \in \mathbb{C}.$$

Theorem 5. *Suppose that f is a transcendental entire function, and that*

$$1 < \lambda < \lambda' < \lambda''.$$

Then there exist $R' > 0$ and a function $\epsilon : \mathbb{R} \to \mathbb{R}$, both of which depend on f, λ, λ' and λ'', such that the following holds. If $r \geq R'$, $n \in \mathbb{N}$, $\eta \geq 0$ and

$$\text{there exists } s \in (\lambda r, \lambda' r) \text{ such that } m(s, f^n) \leq \eta, \tag{2.1}$$

then there exists $w \in B(0, M^n(r,f))$ *such that*

$$f^n(A(r, \lambda'' r)) \supset B(0, M^n(r,f)) \backslash B(w, \epsilon(r) \max\{|w|, \eta\}).$$

Moreover, $\epsilon(r) \to 0$ *as* $r \to \infty$.

To prove Theorem 5 we require some results regarding the maximum modulus of a transcendental entire function. The first two are well-known:

$$\frac{\log M(r,f)}{\log r} \to \infty \text{ as } r \to \infty, \tag{2.2}$$

and

$$\frac{M(cr,f)}{M(r,f)} \to \infty \text{ as } r \to \infty, \quad \text{for } c > 1. \tag{2.3}$$

We require the following [19, Lemma 2.2].

Lemma 2·1. *If f is a transcendental entire function, then there exists* $R_0 = R_0(f) > 0$ *such that, for all* $c > 1$,

$$M(r^c, f) \geq M(r,f)^c, \quad \textit{for } r \geq R_0.$$

We deduce the following by (2.2) and repeated application of Lemma 2·1.

Corollary 2·1. *If f is a transcendental entire function, then there exists* $R_1 = R_1(f) > 0$ *such that, for all* $c > 1$ *and all* $n \in \mathbb{N}$,

$$M^n(r^c, f) \geq M^n(r,f)^c, \quad \textit{for } r \geq R_1.$$

Next, for a transcendental entire function f and for $c > 1$, we define a function

$$\psi_c(r) = \frac{1}{2}\left(\inf_{n \in \mathbb{N}} \frac{M^n(cr,f)}{M^n(r,f)} - 1\right), \quad \text{for } r > 0.$$

We need the following lemma regarding the function ψ_c.

Lemma 2·2. *If f is a transcendental entire function and* $c > 1$, *then*

$$\psi_c(r) \to \infty \textit{ as } r \to \infty.$$

Proof. By Corollary 2·1 and by (2.2), there exists $R = R(f) > 0$ such that, for all $n \in \mathbb{N}$ and $c > 1$,

$$\frac{M^n(cr,f)}{M^n(r,f)} \geq \frac{M^n(r,f)^{1+\log c/\log r}}{M^n(r,f)} = M^n(r,f)^{\log c/\log r} \geq M(r,f)^{\log c/\log r}, \quad \text{for } r \geq R.$$

By a second application of (2.2) we see that $M(r,f)^{\log c/\log r} \to \infty$ as $r \to \infty$. The result follows. □

We also need the following, which is an immediate consequence of [21, Theorem 2.5].

Lemma 2·3. *If f is a transcendental entire function and* $d > 1$, *then there exists* $R_2 = R_2(f,d) > 0$ *such that,*

$$M(dr, f^n) \geq M^n(r,f), \quad \textit{for } r \geq R_2, \ n \in \mathbb{N}.$$

We also require some facts about the hyperbolic metric. If G is a hyperbolic domain and $z_1, z_2 \in G$, then we denote the density of the hyperbolic metric in G at z_1 by $\rho_G(z_1)$, and the hyperbolic distance between z_1 and z_2 in G by $[z_1, z_2]_G$. By [9, Theorem 9.13], there is an absolute constant $C > 1$ such that, with a suitable normalization of hyperbolic density, we have

$$\rho_{\mathbb{C}\backslash\{0,1\}}(z) \geq \frac{1}{2|z|\log(C|z|)}, \quad \text{for } z \in \mathbb{C}\backslash\{0,1\}. \tag{2.4}$$

For each $\tau > 1$, we define a constant $D_\tau > 1$ such that

$$\frac{1}{2}\log D_\tau = \max_{\{z,z':|z|=|z'|=1\}} [z,z']_{A(1/\tau,\tau)}. \tag{2.5}$$

We now prove Theorem 5.

Proof of Theorem 5 Suppose that f is a transcendental entire function and also that $1 < \lambda < \lambda' < \lambda''$. Set

$$\tau = \min\left\{\lambda, \frac{\lambda''}{\lambda'}\right\} > 1 \quad \text{and} \quad c = \frac{1+\lambda}{2} > 1.$$

By Lemma 2·2, we can choose R' sufficiently large that $\psi_c(r) > C^{D_\tau - 1}$, for $r \geq R'$. We also assume that $R' \geq R_2$, where $R_2 = R_2(f,d)$ is the constant from Lemma 2·3 with $d = \frac{\lambda}{c} > 1$.

Define the function $\epsilon : \mathbb{R} \to \mathbb{R}$ by

$$\epsilon(r) = \frac{2C^{1-1/D_\tau}}{\psi_c(r)^{1/D_\tau}}, \quad \text{for } r \geq R'. \tag{2.6}$$

Note that, since $C > 1$ and $D_\tau > 1$,

$$2/\epsilon(r) < \psi_c(r), \quad \text{for } r \geq R'. \tag{2.7}$$

Note also, by Lemma 2·2, that $\epsilon(r) \to 0$ as $r \to \infty$.

Suppose that there exists $r \geq R'$, $n \in \mathbb{N}$, $\eta \geq 0$ and $s \in (\lambda r, \lambda' r)$ such that $m(s, f^n) \leq \eta$. We choose ζ and ζ' such that $|\zeta| = |\zeta'| = s$, and, by Lemma 2·3,

$$|f^n(\zeta)| \leq \eta \text{ and } |f^n(\zeta')| = M(s, f^n) \geq M^n(cr, f). \tag{2.8}$$

Set $A = A(s/\tau, s\tau) \subset A(r, \lambda'' r)$. Suppose, by way of contradiction, that $f^n|_A$ omits values $w_1, w_2 \in B(0, M^n(r,f))$ such that

$$|w_2 - w_1| = \beta \max\{|w_1|, \eta\} \text{ where } \beta \geq \epsilon(r). \tag{2.9}$$

It follows by the contraction of the hyperbolic metric [6, Theorem 4.1] that

$$[\zeta, \zeta']_A \geq [f^n(\zeta), f^n(\zeta')]_{f^n(A)} > [\phi(f^n(\zeta)), \phi(f^n(\zeta'))]_{\mathbb{C}\backslash\{0,1\}}, \tag{2.10}$$

where $\phi(w) = (w - w_1)/(w_2 - w_1)$. By (2.8) and (2.9) we have

$$|\phi(f^n(\zeta))| \leq \frac{|f^n(\zeta)| + |w_1|}{|w_2 - w_1|} \leq \frac{\eta + |w_1|}{\beta \max\{|w_1|, \eta\}} \leq \frac{2}{\beta}$$

and

$$|\phi(f^n(\zeta'))| \geq \frac{|f^n(\zeta')| - |w_1|}{|w_2| + |w_1|} \geq \frac{M^n(cr,f) - M^n(r,f)}{2M^n(r,f)} \geq \psi_c(r).$$

Note that, by (2.7) and (2.9), we have that $\psi_c(r) > 2/\beta$. Suppose first that $2/\beta \geq 1$. Then, by (2.4), (2.5) and (2.10) we deduce that

$$\frac{1}{2}\log D_\tau > \int_{2/\beta}^{\psi_c(r)} \frac{dt}{2t\log(Ct)} = \frac{1}{2}\log\frac{\log(C\psi_c(r))}{\log(2C/\beta)},$$

and hence $\beta < \epsilon(r)$, which is a contradiction. On the other hand, if $2/\beta < 1$, then we deduce similarly that

$$\frac{1}{2}\log D_\tau > \int_1^{\psi_c(r)} \frac{dt}{2t\log(Ct)} = \frac{1}{2}\log\frac{\log(C\psi_c(r))}{\log C},$$

which is a contradiction to our choice of R'. □

In our proof of Theorem 2, we use the following immediate corollary of Theorem 5. This is a generalisation of [15, Corollary 2.3], which considered the case $n = 1$. Here we use the following notation for a closed annulus

$$\overline{A}(r_1, r_2) = \{z : r_1 \leq |z| \leq r_2\}, \quad \text{for } 0 < r_1 < r_2.$$

COROLLARY 2·2. *Suppose that f is a transcendental entire function. Then there exists $R_3 = R_3(f) > 0$ such that the following holds. If there exists $r \geq R_3$, $n \in \mathbb{N}$ and $s \in (2r, 4r)$ such that $m(s, f^n) \leq 1$, and S, S', T, T' satisfy*

$$2 < S < S', \ T < T' < M^n(r, f) \text{ and } S' \leq \frac{1}{2}T,$$

then $f^n(A(r, 8r))$ contains $\overline{A}(S, S')$ or $\overline{A}(T, T')$.

3. *Properties of $A'(f)$ and $A''(f)$*

In this section we prove Theorem 1 and Theorem 2.

Proof of Theorem 1 Adapting the notation in [26], let $\mathcal{M}(f)$ be the set of points where a transcendental entire function, f, attains its maximum modulus; that is,

$$\mathcal{M}(f) = \{z : |f(z)| = M(|z|, f)\}. \tag{3.1}$$

It is well-known – see, for example, [27, Theorem 10] – that $\mathcal{M}(f)$ consists of, at most, a countable union of *maximal curves,* which are analytic except possibly at their endpoints.

If $z \in A'(f)$, where f is a transcendental entire function, then, by (1.2), there exists $N \in \mathbb{N}$ such that

$$f^n(z) \in \mathcal{M}(f), \quad \text{for } n \geq N. \tag{3.2}$$

In particular, it follows that,

$$A'(f) \subset \bigcup_{k=0}^{\infty} f^{-k}(\mathcal{M}(f)).$$

Hence $A'(f)$ is contained in a countable union of curves, which are analytic except possibly at their endpoints, as required. □

In order to prove Theorem 2 we require a number of preliminary results, the first of which is a version of [20, Lemma 1], which considered images of sets under a single iteration of f.

LEMMA 3·1. *Suppose that $(E_n)_{n\in\mathbb{N}}$ is a sequence of compact sets and $(m_n)_{n\in\mathbb{N}}$ is a sequence of integers. Suppose also that f is a transcendental entire function such that $E_{n+1} \subset f^{m_n}(E_n)$, for $n \in \mathbb{N}$. Set $p_n = \sum_{k=1}^n m_k$, for $n \in \mathbb{N}$. Then there exists $\zeta \in E_1$ such that*

$$f^{p_n}(\zeta) \in E_{n+1}, \quad \textit{for } n \in \mathbb{N}. \tag{3.3}$$

If, in addition, $E_n \cap J(f) \neq \emptyset$, for $n \in \mathbb{N}$, then there exists $\zeta \in E_1 \cap J(f)$ such that (3.3) holds.

Proof. For $n \in \mathbb{N}$, we set

$$F_n = \{z \in E_1 : f^{p_1}(z) \in E_2, f^{p_2}(z) \in E_3, \ldots, f^{p_n}(z) \in E_{n+1}\}.$$

It follows by hypothesis that $(F_n)_{n\in\mathbb{N}}$ is a decreasing sequence of non-empty compact subsets of E_1, and so $F = \bigcap_{k=1}^{\infty} F_k$ is a non-empty subset of E_1. We choose $\zeta \in F$ and the result follows.

Since $J(f)$ is completely invariant, the second statement follows by applying the first statement to the non-empty compact sets $E_n \cap J(f)$, for $n \in \mathbb{N}$. □

We require some results concerning multiply connected Fatou components. The first is the following well-known result of Baker [2, Theorem 3.1]. We say that a set U surrounds a set V if V is contained in a bounded component of $\mathbb{C}\backslash U$. If U is a Fatou component, we write U_n, $n \geq 0$, for the Fatou component containing $f^n(U)$. We also let $\operatorname{dist}(z, U) = \inf_{w\in U} |z - w|$, for $z \in \mathbb{C}$.

Lemma 3·2. *Suppose that f is a transcendental entire function and that U is a multiply connected Fatou component of f. Then each U_n is bounded and multiply connected, U_{n+1} surrounds U_n for large n, and* $\operatorname{dist}(0, U_n) \to \infty$ as $n \to \infty$.

In addition we require [21, Theorem 4.4].

Lemma 3·3. *Suppose that f is a transcendental entire function, and that U is a multiply connected Fatou component of f. Then $\overline{U} \subset A(f)$.*

We also require some notation and results from [25]. Define a function R_A by

$$R_A(z) = \max\{R \geq 0 : M^n(R, f) \to \infty \text{ as } n \to \infty \text{ and } |f^n(z)| \geq M^n(R, f), \text{for } n \in \mathbb{N}\}, \tag{3.4}$$

where we set $R_A(z) = -1$ if the set on the right-hand side of (3.4) is empty. If the set on the right-hand side of (3.4) is not empty, then the existence of a maximum follows from the continuity of $M(R, f)$.

If U is a multiply connected Fatou component which surrounds the origin, then we define $\partial_{\text{int}} U$ as the boundary of the component of $\mathbb{C}\backslash\overline{U}$ that contains the origin. We use the following lemma, which is a combination of [25, Lemma 4.2(c)] and [25, Lemma 4.4(a)].

Lemma 3·4. *Suppose that f is a transcendental entire function. Then there exists $R_4 = R_4(f) > 0$ such that the following holds. Suppose that U is a multiply connected Fatou component of f, which surrounds the origin and which satisfies* $\operatorname{dist}(0, U) \geq R_4$. *There there exists $R' \geq 0$ such that*

$$R_A(z) = R', \quad \text{for } z \in \partial_{\text{int}} U.$$

We also need the following simple result.

Lemma 3·5. *Suppose that f is a transcendental entire function, that $z \in A'(f)$ and that $R_A(z) \geq 0$. Then there exists $N \in \mathbb{N}$ such that*

$$|f^n(z)| = M^n(R_A(z), f), \quad \text{for } n \geq N.$$

Proof. Suppose that $z \in A'(f)$. It follows from (1.2) that there exists $N \in \mathbb{N}$ such that, with $R = |f^{N-1}(z)|$, we have $M^n(R, f) \to \infty$ as $n \to \infty$, and

$$|f^n(z)| = M^{n+1-N}(R, f), \quad \text{for } n \geq N.$$

Since $R_A(z) \geq 0$ we have, by (3.4), that

$$M^{N-1}(0, f) \leq M^{N-1}(R_A(z), f) \leq |f^{N-1}(z)| = R.$$

It follows, by the continuity of the function M, that there exists $R' \geq R_A(z)$ such that $M^{N-1}(R', f) = R$. We deduce that

$$|f^n(z)| = M^n(R', f), \quad \text{for } n \geq N.$$

The result follows since, by (3.4), we must have $R_A(z) = R'$. □

We now prove Theorem 2.

Proof of Theorem 2 Let f be a transcendental entire function. Our proof splits into two cases. If f has a multiply connected Fatou component, then we show that there are, at most, countably many points in $A'(f) \cap J(f)$ on the inner boundary of some multiply connected Fatou component, which is a continuum in $A(f) \cap J(f)$. If f has no multiply connected Fatou component, then we use Theorem 5 to construct a point ζ which lies in $A''(f) \cap J(f)$.

Suppose first that f has a multiply connected Fatou component, U. By Lemma 3·2 we may assume that U surrounds the origin and that $\operatorname{dist}(0, U) \geq R_4$, where R_4 is the constant from Lemma 3·4. It follows from Lemma 3·4 that there exists $R' \geq 0$ such that $R_A(z) = R'$, for $z \in \partial_{\text{int}} U$. We deduce by Lemma 3·5 that for each $z \in \partial_{\text{int}} U \cap A'(f)$ there exists $N = N(z) \in \mathbb{N}$ such that $|f^n(z)| = M^n(R', f)$, for $n \geq N$.

We recall the definition of the set $\mathcal{M}(f)$ in (3.1). It follows, by (3.2), that

$$\partial_{\text{int}} U \cap A'(f) \subset \bigcup_{k=0}^{\infty} f^{-k} \left(\{ z \in \mathcal{M}(f) : |z| = M^k(R', f) \} \right). \tag{3.5}$$

It is known [27, Theorem 10] that, for each $R \geq 0$, the set $\{z \in \mathcal{M}(f) : |z| = R\}$ is finite. Hence the right-hand side of (3.5) is countable. Since, by Lemma 3·3, $\partial_{\text{int}} U$ is a continuum contained in $A(f) \cap J(f)$, we deduce that $\partial_{\text{int}} U \cap A''(f) \cap J(f)$ is uncountable.

On the other hand, suppose that f has no multiply connected Fatou component. Set $K = \frac{1}{2048}$ and let R_3 be the constant from Corollary 2·2. (We note that this choice of K is smaller than necessary for the proof of Theorem 2, but facilitates the proof of Lemma 5·2; see equations (3.11) and (5.16) below.) Define a function

$$\mu(r) = KM(r, f), \quad \text{for } r > 0.$$

We construct an increasing sequence of real numbers $(r_n)_{n \in \mathbb{N}}$ and a sequence of integers $(m_n)_{n \in \mathbb{N}}$. Choose $r_1 \geq R_3$ sufficiently large that $B(0, r_1) \cap J(f) \neq \emptyset$ and also, by (2.2), that

$$\mu(r) > \max\{r^2, 2\}, \quad \text{for } r \geq r_1. \tag{3.6}$$

By (2.3), we may also assume that r_1 is sufficiently large that

$$\mu(r) \geq \frac{1}{K} M(Kr, f), \quad \text{for } r \geq r_1.$$

We deduce that

$$\mu^n(r) \geq M^n\left(Kr, f\right), \quad \text{for } r \geq r_1,\ n \in \mathbb{N}. \tag{3.7}$$

The construction proceeds inductively. Suppose that, for some $k \geq 1$, we have constructed the sequences $(r_n)_{n \leq k}$ and $(m_n)_{n<k}$. Let A be the annulus $A = A(r_k, 8r_k)$ and let A' be the annulus $A' = A(2r_k, 4r_k)$. Suppose that

$$m(s, f^n) > 1, \quad \text{for } s \in (2r_k, 4r_k),\ n \in \mathbb{N}.$$

Then, by Montel's theorem, $\{f^n\}_{n\in\mathbb{N}}$ is a normal family in A', and so $A' \subset F(f)$. Hence, by the choice of r_1, A' is contained in a multiply connected Fatou component of f, which a contradiction. Therefore, there exists $m_k \in \mathbb{N}$ and $s \in (2r_k, 4r_k)$ such that $m(s, f^{m_k}) \leq 1$.

Set $S = KM^{m_k}(r_k, f)$, $S' = 8S$, $T = 16S$ and $T' = 128S$. It follows from Corollary 2·2 that $f^{m_k}(A(r_k, 8r_k))$ contains either $\overline{A}(S, S')$, in which case we set $r_{k+1} = S$, or $\overline{A}(T, T')$, in which case we set $r_{k+1} = T$. Note that in either case we have, by (3.6), that

$$r_{k+1} \geq KM^{m_k}(r_k, f) \geq \mu^{m_k}(r_k) > r_k.$$

This completes the construction of the sequences.

We now define a sequence of closed annuli $(E_n)_{n\in\mathbb{N}}$ by

$$E_n = \overline{A}(r_n, 8r_n), \quad \text{for } n \in \mathbb{N}.$$

It follows at once from the above construction that $E_{n+1} \subset f^{m_n}(E_n)$, for $n \in \mathbb{N}$. We also have, by the choice of r_1 and since f has no multiply connected Fatou component, that $E_n \cap J(f) \neq \emptyset$, for $n \in \mathbb{N}$. Hence, by Lemma 3·1, there exists $\zeta \in J(f)$ such that (3.3) holds with $p_n = \sum_{k=1}^n m_k$, for $n \in \mathbb{N}$.

We claim next that $\zeta \in A(f)$. We note, by (3.3) and by construction, that

$$\begin{aligned} |f^{p_n}(\zeta)| &\geq r_{n+1} \\ &\geq KM^{m_n}(r_n, f) \\ &\geq KM^{m_n}(KM^{m_{n-1}}(r_{n-1}, f), f) \\ &\ \ . \\ &\ \ . \\ &\ \ . \\ &\geq \mu^{p_n}(r_1), \quad \text{for } n \in \mathbb{N}. \end{aligned}$$

By (3.6), there exists $\ell \in \mathbb{N}$ such that $\mu^\ell(r_1) \geq r_1/K$. We deduce by (3.7) that

$$\mu^{n+\ell}(r_1) \geq \mu^n\left(\frac{r_1}{K}\right) \geq M^n(r_1, f), \quad \text{for } n \in \mathbb{N}. \tag{3.8}$$

Hence

$$|f^{p_n}(\zeta)| \geq \mu^{p_n}(r_1) \geq M^{p_n-\ell}(r_1, f), \quad \text{for sufficiently large values of } n.$$

It follows that $|f^k(\zeta)| \geq M^{k-\ell}(r_1, f)$, for sufficiently large values of k, and so $\zeta \in A(f)$ as claimed.

Finally, suppose that $\zeta \in A'(f)$, in which case, by (1.2), there exists $N \in \mathbb{N}$ such that

$$|f^{n+p}(\zeta)| = M^n(|f^p(\zeta)|, f), \quad \text{for } n \in \mathbb{N} \text{ and } p \geq N. \tag{3.9}$$

However, by the choices of S, T and K, we have that $r_{n+1} \leq \frac{1}{128}M^{m_n}(r_n, f)$, for

$n \in \mathbb{N}$, and so if $z \in E_n$ and $f^{m_n}(z) \in E_{n+1}$, for some $n \in \mathbb{N}$, then

$$|f^{m_n}(z)| \leq 8r_{n+1} \leq \frac{1}{16} M^{m_n}(r_n, f) \leq \frac{1}{16} M^{m_n}(|z|, f). \tag{3.10}$$

Hence, by (3.3) and (3.10) we have, for all sufficiently large $n \in \mathbb{N}$,

$$|f^{m_n + p_{n-1}}(\zeta)| = |f^{m_n}(f^{p_{n-1}}(\zeta))| \leq \frac{1}{16} M^{m_n}(|f^{p_{n-1}}(\zeta)|, f). \tag{3.11}$$

This is in contradiction to (3.9). We deduce that $\zeta \in A''(f) \cap J(f)$, as required. □

4. *Proof of Theorem* 3

We require the following well-known distortion lemma; see, for example, [3, Lemma 7].

Lemma 4·1. *Suppose that f is a transcendental entire function and that $U \subset I(f)$ is a simply connected Fatou component of f. Suppose that K is a compact subset of U. Then there exist $C > 1$ and $N \in \mathbb{N}$ such that*

$$\frac{1}{C}|f^n(z)| \leq |f^n(w)| \leq C|f^n(z)|, \quad \text{for } w, z \in K, n \geq N.$$

We also require the following [20, Lemma 10]. Here a set S is *backwards invariant* if $z \in S$ implies that $f^{-1}(\{z\}) \subset S$.

Lemma 4·2. *Suppose that f is a transcendental entire function, and that $E \subset \mathbb{C}$ is non-empty.*

(a) If E is backwards invariant, contains at least three points and $\operatorname{int} E \cap J(f) = \emptyset$, then $J(f) \subset \partial E$.

(b) If every component of $F(f)$ that meets E is contained in E, then $\partial E \subset J(f)$.

We now prove Theorem 3, and so we suppose that f is a transcendental entire function. For part (a) of the theorem, let U be a simply connected Fatou component of f which meets $A(f)$. Note [21, Theorem 1.2] that $U \subset A(f)$. Suppose, by way of contradiction, that $z_0 \in U \cap A'(f)$. Then there exists $N \in \mathbb{N}$, a point $z_N = f^N(z_0)$ and $R = |z_N|$ such that

$$|f^n(z_N)| = M^n(R, f), \text{ for } n \in \mathbb{N}.$$

Let U_N be the Fatou component of f containing z_N, and note (see, for example, [22, Lemma 4.2]) that U_N is also simply connected. Choose a point $z'_N \in U_N$ such that $|z'_N| = R' < R$. Clearly

$$|f^n(z'_N)| \leq M^n(R', f), \text{ for } n \in \mathbb{N}.$$

Since there exists $N' \in \mathbb{N}$ such that $f^n(z'_N) \neq 0$, for $n \geq N'$, we deduce that

$$\frac{|f^n(z_N)|}{|f^n(z'_N)|} \geq \frac{M^n(R, f)}{M^n(R', f)}, \quad \text{for } n \geq N'. \tag{4.1}$$

Let $K = \{z_N, z'_N\}$. We deduce by Lemma 4·1 that the left-hand side of (4.1) is bounded for $n \geq N'$. However, it follows from Corollary 2·1 that the right-hand side of (4.1) tends to infinity as $n \to \infty$. This contradiction completes the proof of part (a) of the theorem.

Suppose next that U is a multiply connected Fatou component of f. Part (b) of the theorem is an immediate consequence of Lemma 3·3 and Theorem 1.

For part (c) of the theorem, we note that if follows from Theorem 2 that $A''(f) \cap J(f)$ is an infinite set, since for each $z \in A''(f) \cap J(f)$ at least one of the points $z, f(z)$ or

$f^2(z)$ must have infinitely many preimages. It is known [3, Theorem 4] that the set of repelling periodic points of f is dense in $J(f)$. Clearly $A''(f)$ contains no periodic points, and so $\operatorname{int} A''(f) \subset F(f)$. We deduce by Lemma 4·2(a), applied with $E = A''(f) \cap J(f)$ and then with $E = A''(f)$, that $A''(f)$ is dense in $J(f)$, and that $J(f) \subset \partial A''(f)$.

For part (d) of the theorem, suppose that $A'(f) \cap J(f) \neq \emptyset$. For the same reasons as above, it follows that $A'(f) \cap J(f)$ is an infinite set and $\operatorname{int} A'(f) \subset F(f)$. We deduce by Lemma 4·2(a), applied with $E = A'(f) \cap J(f)$ and then with $E = A'(f)$, that $A'(f)$ is dense in $J(f)$, and that $J(f) \subset \partial A'(f)$.

For part (e) of the theorem, suppose that $A'(f) \cap F(f) = \emptyset$. We deduce from part (a), Lemma 3·3 and Lemma 4·2(b) that $\partial A''(f) \subset J(f)$, and hence by part (c) that $J(f) = \partial A''(f)$.

Finally, for part (f) of the theorem, suppose that $A'(f) \cap J(f) \neq \emptyset$ and also that $A'(f) \cap F(f) = \emptyset$. We deduce from Lemma 4·2(b) that $\partial A'(f) \subset J(f)$, and hence by part (d) that $J(f) = \partial A'(f)$.

5. *Examples*

EXAMPLE 1. Let $f_1(z) = \exp(z)$. We observe that $\mathcal{M}(f_1) = [0, \infty)$ and also that $f_1(\mathcal{M}(f_1)) \subset \mathcal{M}(f_1)$. We deduce that $A'(f_1)$ is the countable union of analytic curves

$$A'(f) = \bigcup_{k=0}^{\infty} f_1^{-k}([0, \infty)).$$

Since [3, Lemma 4] the set $\bigcup_{k=0}^{\infty} f_1^{-k}(-1)$ is dense in $J(f_1)$ and, as is well-known, $I(f_1) \subset J(f_1)$, we deduce that $\bigcup_{k=0}^{\infty} f_1^{-k}(-1)$ is dense in $A'(f_1)$. Thus

$$\bigcup_{k=0}^{\infty} f_1^{-k}((-\infty, 0)) \subset A'(f_1) \subset \overline{\bigcup_{k=0}^{\infty} f_1^{-k}((-\infty, 0))}.$$

Rempe [14, Proposition 3.1] showed that the set $\bigcup_{k=0}^{\infty} f_1^{-k}((-\infty, 0))$ is connected. We deduce that $A'(f_1)$ is connected.

EXAMPLE 2. Let $f_2(z) = i \exp(z)$. We observe that $\mathcal{M}(f_2) = [0, \infty)$ and also that $f_2(\mathcal{M}(f_2)) \cap \mathcal{M}(f_2) = \emptyset$. It follows that $A'(f_2) = \emptyset$.

EXAMPLE 3. Baker [1] showed that constants $C > 0$ and $0 < a_1 < a_2 < \cdots$ can be chosen in such a way that

$$f_3(z) = Cz^2 \prod_{n=1}^{\infty} \left(1 + \frac{z}{a_n}\right)$$

is a transcendental entire function with a multiply connected Fatou component. Since f_3 only has positive coefficients in its power series, it is clear that $A'(f_3)$ contains an unbounded interval in the positive real axis. We deduce by Lemma 3·2 that we have $A'(f_3) \cap F(f_3) \neq \emptyset$. It follows, by Theorem 3 part (b), that $\partial A''(f_3)$ and $\partial A'(f_3)$ both contain points of $F(f_3)$.

EXAMPLE 4. Let $f_4(z) = i f_3(z)$, where f_3 is the function in Example 3. It can be seen from the construction in [1] that f_4 also has a multiply connected Fatou component. However, for the same reason as the function f_2 in Example 2, we see that $A'(f_4) = \emptyset$.

Example 5. We define a family of transcendental entire functions by

$$g_\alpha(z) = \alpha \exp(e^{z^2} + \sin z), \quad \text{for } \alpha > 0.$$

The function g_1 was considered by Hardy [8]. It is clear that Theorem 4 follows from the following lemmas.

Lemma 5·1. *If $\alpha > 0$, then $A'(g_\alpha)$ is uncountable and totally disconnected.*

Lemma 5·2. *If $\alpha > 0$ is sufficiently small, then there are uncountably many singleton components of $A''(g_\alpha)$. Moreover, $A''(g_\alpha)$ has at least one unbounded component.*

Proof of Lemma 5·1 To show that $A'(g_\alpha)$ is uncountable and totally disconnected we show that $A'(g_\alpha)$ consists of the preimages of a set $S \subset \mathbb{R}$, and we use properties of the maximum modulus of g_α to show that S is uncountable and totally disconnected.

Although Hardy [8] considered only g_1, it follows easily from his result that there exists $r_0 > 0$ such that the following holds. If $|z| \geq r_0$, then $z \in \mathcal{M}(g_\alpha)$ if and only if z is real, and lies on the positive real axis if $\sin |z| > 0$ and on the negative real axis if $\sin |z| < 0$. Since g_α maps points on the real axis to the positive real axis, we deduce that

$$A'(g_\alpha) = \bigcup_{k=0}^{\infty} g_\alpha^{-k}(S) \quad \text{where } S = \{x \geq r_0 : \sin g_\alpha^n(x) \geq 0, \text{ for } n \geq 0\}. \tag{5.1}$$

We prove first that S is uncountable. Let $I_k = [2k\pi, (2k+1)\pi]$, for $k \in \mathbb{N}$. We observe that $g_\alpha'(x) \to \infty$ as $x \to \infty$. Hence there exists $k_0 \geq r_0/2\pi$, $k_0 \in \mathbb{N}$, such that if $k \in \mathbb{N}$ and $k \geq k_0$, then there exists $k' > k$ such that $I_{k'} \cup I_{k'+1} \subset g_\alpha(I_k)$.

We construct an increasing sequence of integers $(k_n)_{n\in\mathbb{N}}$ as follows, noting that at each stage in the construction we have two distinct choices. First set either $k_1 = k_0$ or $k_1 = k_0 + 1$. Suppose inductively that k_n is defined for some $n \in \mathbb{N}$. There exists $k_n' > k_n$ such that $I_{k_n'} \cup I_{k_n'+1} \subset g_\alpha(I_{k_n})$. We choose either $k_{n+1} = k_n'$ or $k_{n+1} = k_n' + 1$. This completes the construction of the sequence. We then find a point $\zeta \in S$ by Lemma 3·1, with $E_n = I_{k_n}$ and $m_n = 1$, for $n \in \mathbb{N}$.

Now let S' be the subset of S consisting of points which can be constructed in this way. Since at each stage in the construction we had two distinct choices, there is a surjection from S' to the set of infinite binary strings. Hence S', and so also S, is uncountable.

Next we prove that S is totally disconnected. Let $J_k = [(2k+5/4)\pi, (2k+7/4)\pi]$, for $k \in \mathbb{N}$. It follows from a construction similar to the above that if $k \geq r_0/2\pi$ is sufficiently large, then J_k contains uncountably many points in $\mathbb{R}\backslash S$ and indeed in $\mathbb{R}\backslash A'(g_\alpha)$. (We simply replace all references to the intervals I_n in the construction above to the intervals J_n, and thereby construct uncountably many points whose orbit lies in $\cup_{n\in\mathbb{N}} J_n$, in which case each point lies in $\mathbb{R}\backslash A'(g_\alpha)$.)

Suppose next that $x_1 < x_2$ are points in S. Since g_α has a large derivative on the real axis, we deduce that there exists an arbitrarily large k, and $N \in \mathbb{N}$ such that $g_\alpha^N(x_1) < (2k+5/4)\pi$ and $(2k+7/4)\pi < g_\alpha^N(x_2)$. Hence, there is a point $x' \in \mathbb{R}\backslash S$ such that $g_\alpha^N(x_1) < x' < g_\alpha^N(x_2)$. Since g_α is increasing on $\mathbb{R}$, there is a point x'' such that $x_1 < x'' < x_2$ and $g_\alpha^N(x'') = x'$. It follows that $x'' \notin S$, and we deduce that S is totally disconnected, as required.

It is known ([13, Lemma 2.5] and see also [11, Chapter II]) that a countable union of compact, totally disconnected subsets of $\mathbb{C}$ is totally disconnected. We deduce from this result, from the fact that S is a countable union of compact, totally disconnected sets, and from (5.1) that $A'(g_\alpha)$ is indeed totally disconnected. $\square$

Proof of Lemma 5·2 Consider first the real-valued function $x \to g_\alpha(x)$. It is straightforward to show that $g''_\alpha(x) > 0$, for $x > 0$. We may assume, therefore, that $\alpha > 0$ is sufficiently small that $g_\alpha(x)$ has exactly two fixed points. An elementary calculation shows that there is an attracting fixed point $p_\alpha \in (0, 1)$, and a repelling fixed point $q_\alpha \in (p_\alpha, \infty)$.

Suppose that $x \in (q_\alpha, \infty)$. Then there exists $N = N(x) \in \mathbb{N}$ such that, by the result of Hardy [8] mentioned in the proof of Lemma 5·1,

$$g^{n+1}_\alpha(x) \geq \frac{1}{e^2} \max\{g_\alpha(g^n_\alpha(x)), g_\alpha(-g^n_\alpha(x))\} = \frac{1}{e^2} M(g^n_\alpha(x), g_\alpha), \quad \text{for } n \geq N. \tag{5.2}$$

It follows by [21, Theorem 2.9] that $x \in A(g_\alpha)$. Hence $(q_\alpha, \infty) \subset A(g_\alpha)$.

It was shown in the proof of Lemma 5·1 that, for sufficiently large values of $k \in \mathbb{N}$, $[(2k+5/4)\pi, (2k+7/4)\pi] \backslash A'(g_\alpha)$ is uncountable. We deduce that $(q_\alpha, \infty) \cap A''(g_\alpha)$ is uncountable. We claim that $(q_\alpha, \infty) \cap A''(g_\alpha)$ is totally disconnected. Suppose that $x_1 < x_2$ are points in $(q_\alpha, \infty) \cap A''(g_\alpha)$. Arguing as in the proof of Lemma 5·1, we deduce that there exists an arbitrarily large k, and $N \in \mathbb{N}$ such that $g^N_\alpha(x_1) < 2k\pi$ and $(2k+1)\pi < g^N_\alpha(x_2)$. It follows that there is a point $x' \in A'(g_\alpha)$ such that $g^N_\alpha(x_1) < x' < g^N_\alpha(x_2)$. Since g_α is increasing on $\mathbb{R}$, there is a point x'' such that $x_1 < x'' < x_2$ and $g^N_\alpha(x'') = x'$. It follows that $x'' \notin A''(g_\alpha)$, and we deduce that $(q_\alpha, \infty) \cap A''(g_\alpha)$ is totally disconnected, as claimed.

We now show that there exists $\alpha_0 > 0$ such that

$$(q_\alpha, \infty) \text{ is a component of } A(g_\alpha), \quad \text{for } 0 < \alpha < \alpha_0. \tag{5.3}$$

Since $(q_\alpha, \infty) \cap A''(g_\alpha)$ is uncountable and totally disconnected, the fact that $A''(g_\alpha)$ has uncountably many singleton components, for $0 < \alpha < \alpha_0$, then follows from (5.3).

The proof of (5.3) is complicated. Roughly speaking, our method is as follows. First we choose α sufficiently small that $F(g_\alpha)$ has an unbounded attracting basin U, which lies outside the escaping set and contains infinitely many preimages of the positive imaginary axis. We then deduce (5.3) by a careful analysis of the properties of these preimages, and from the properties of the fast escaping set.

First we note some facts about the function g_α. The proof is simplified slightly by noting that g_α satisfies the equation

$$g_\alpha(\overline{z}) = \overline{g_\alpha(z)}, \quad \text{for } z \in \mathbb{C}. \tag{5.4}$$

We observe that if $z = x + iy$ and $g_\alpha(z) = Re^{i\theta} = u + iv$, then, by a calculation,

$$\log R = e^{x^2-y^2} \cos 2xy + \sin x \cosh y + \log \alpha, \tag{5.5}$$

$$\theta = e^{x^2-y^2} \sin 2xy + \cos x \sinh y, \tag{5.6}$$

$$\begin{aligned}\frac{\partial u}{\partial x} &= u(e^{x^2-y^2}(2x \cos 2xy - 2y \sin 2xy) + \cos x \cosh y \\ &\quad - \tan \theta (e^{x^2-y^2}(2x \sin 2xy + 2y \cos 2xy) - \sin x \sinh y)),\end{aligned} \tag{5.7}$$

$$\begin{aligned}\frac{\partial v}{\partial y} &= v(e^{x^2-y^2}(-2y \cos 2xy - 2x \sin 2xy) + \sin x \sinh y \\ &\quad + \cot \theta (e^{x^2-y^2}(-2y \sin 2xy + 2x \cos 2xy) + \cos x \cosh y)).\end{aligned} \tag{5.8}$$

Let $\Gamma_0 = \{z : \arg z = \pi/2\}$ and $\gamma_0 = \{z : \arg z = \pi/4\}$. We deduce from (5.5) that g_α is bounded on Γ_0.

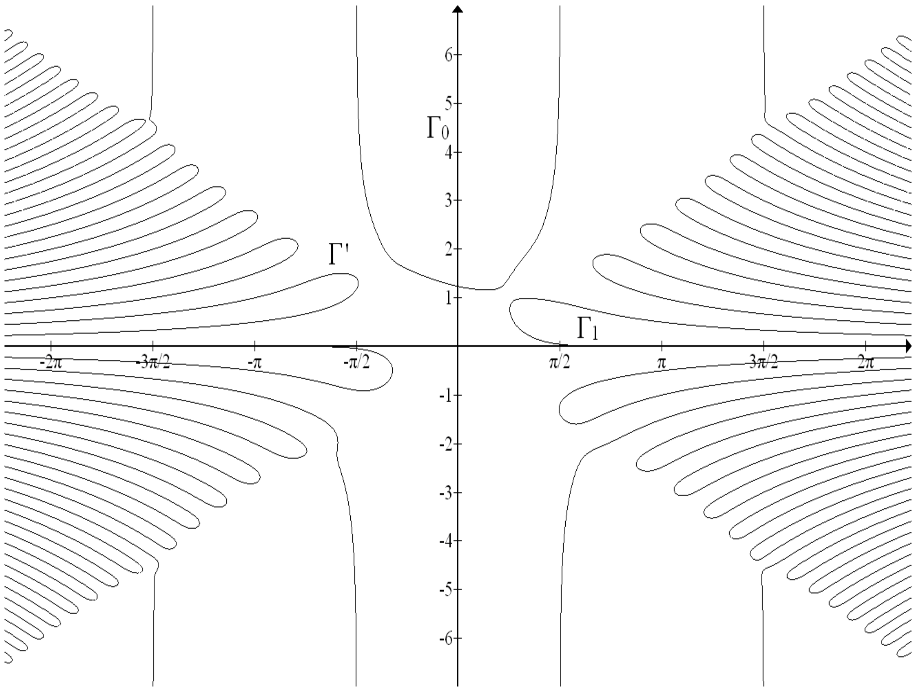

Fig. 1. Some of the preimages of the positive imaginary axis, calculated by solving equation (5.6) for $\theta = \pi/2$.

Next we define three domains $V'' \subset V' \subset V$ in which g_α has certain useful properties. First, for a large value of $K > 1$, which we fix later, let V be the domain

$$V = \{z = x + iy : x > K,\ 0 < y < e^{-x^2}\}.$$

It follows from (5.5) and (5.6) that we can choose K sufficiently large that, with $z = x + iy \in V$,

$$e^{x^2} - 2 \leq \log \frac{|g_\alpha(z)|}{\alpha} \leq e^{x^2} + 2 \quad \text{and} \quad 2y(xe^{x^2} - 1) \leq \arg g_\alpha(z) \leq 2y(xe^{x^2} + 1). \tag{5.9}$$

We deduce that V has unbounded intersection with at least one component of $g_\alpha^{-1}(\Gamma_0)$. Let Γ_1 be the intersection of the preimage component of least positive imaginary part with V, and note by (5.9) that

$$y \sim \frac{\pi}{4x} e^{-x^2} \text{ as } x \to \infty, \quad \text{for } x + iy \in \Gamma_1.$$

It follows by differentiating (5.6) that, increasing the size of K if necessary, we may assume that

$$\frac{dy}{dx} < 0, \quad \text{for } x + iy \in \Gamma_1. \tag{5.10}$$

We deduce also that V has unbounded intersection with at least one component of $g_\alpha^{-1}(\gamma_0)$. Let γ_1 be the intersection of the preimage component of least positive imaginary part with V, and note by (5.9) that

$$y \sim \frac{\pi}{8x} e^{-x^2} \text{ as } x \to \infty, \quad \text{for } x + iy \in \gamma_1.$$

Increasing the size of K if necessary, we may assume that Γ_1 and γ_1 each intersect the boundary of V only at a point with real part K and modulus less than $2K$. We may also assume that if $K < x < x'$, then

$$\text{if } x + iy \in \gamma_1, \text{ then there exists } x' + iy' \in \Gamma_1 \text{ such that } y' < 4y. \tag{5.11}$$

Let $V' \subset V$ be the domain bounded by Γ_1, $\{z : \text{Re}(z) = K\}$, and the real axis. Increasing the size of K again, if necessary, we may assume, by (5.9), that

$$0 < \arg g_\alpha(z) < 3\pi/4, \quad \text{for } z \in V'. \tag{5.12}$$

Then let $V'' \subset V'$ be the domain bounded by γ_1, $\{z : \text{Re}(z) = K\}$, and the real axis. We deduce from (5.7) and (5.8) that if $x + iy \in V'' \cup \gamma_1$ and $g_\alpha(x + iy) = u + iv$, then as $x \to \infty$,

$$\frac{\partial u}{\partial x} = u\left(2xe^{x^2} + O(x)\right) \quad \text{and} \quad \frac{\partial v}{\partial y} = v\left(2xe^{x^2} \cot(\arg g_\alpha(z)) + O(x)\right).$$

Hence, by (5.9), may assume that

$$\frac{\partial}{\partial x} \text{Re}(g_\alpha(x + iy)) > 0 \quad \text{and} \quad \frac{\partial}{\partial y} \text{Im}(g_\alpha(x + iy)) > 0, \quad \text{for } x + iy \in V'' \cup \gamma_1. \tag{5.13}$$

Finally, by a similar argument to the one given earlier relating to Γ_1, we note that there is a component Γ' of $g_\alpha^{-1}(\Gamma_0)$ which is asymptotic to the negative real axis. We omit the details. Increasing K one last time, if necessary, we may assume that $B(0, K) \cap \Gamma' \neq \emptyset$. Note that Γ_1, γ_1 and Γ' are each independent of α.

Next we fix a value of α_0. We choose $0 < \alpha_0 < e^{-1}$ sufficiently small that if $0 < \alpha < \alpha_0$, then g_α has the two fixed points p_α and q_α discussed earlier, and also that

$$g_\alpha\left(B(0, 2K) \cup \Gamma_0\right) \subset B(0, 1).$$

Suppose then that $0 < \alpha < \alpha_0$. We deduce that g_α has an unbounded simply connected Fatou component, U, which contains Γ_0, the disc $B(0, 2K)$ and so also Γ_1 and Γ', the attracting fixed point p_α, and indeed the interval $(0, q_\alpha)$. Note that $q_\alpha \geq 2K$, but we do not assume that equality holds. Clearly $U \cap A(g_\alpha) = \emptyset$, and so all preimages of Γ_0 lie outside $A(g_\alpha)$.

Clearly $q_\alpha \in J(g_\alpha) \cap \mathbb{C} \backslash A(g_\alpha)$. We claim that $(q_\alpha, \infty) \subset J(g_\alpha)$. Since g_α is bounded on Γ_0, it follows by Lemma 3·2 that g_α has no multiply connected Fatou component.

Moreover, the set $A'(g_\alpha)$ is dense in (q_α, ∞). We deduce by Theorem 3 part (a) that $(q_\alpha, \infty) \subset J(g_\alpha)$, as claimed.

Suppose next that $U' \neq U$ is a component of $F(g_\alpha)$ such that $g_\alpha(U') \subset U$. We claim that $U' \cap V' = \emptyset$. Suppose, to the contrary, that $U' \cap V' \neq \emptyset$. Since the boundary of V' consists of points either in U or in $J(g_\alpha)$, we deduce that $U' \subset V'$. Now [10, Theorem 3] $g_\alpha(U')$ and U may differ by at most two points. However, $\Gamma' \subset U$ contains points with argument arbitrarily close to π. This is a contradiction, by (5.12), completing the proof of our claim.

For $\xi > K$, let $y(\xi)$ be such that $\xi + iy(\xi)$ is the point on γ_1 of real part ξ; this point is unique by (5.13). By (5.9), we can choose $\xi > q_\alpha$ sufficiently large that

$$\operatorname{Re}(g_\alpha(\xi + iy')) > 2\xi \quad \text{and} \quad \operatorname{Im}(g_\alpha(\xi + iy')) > 8y', \quad \text{for } y' \in (0, y(\xi)]. \tag{5.14}$$

We claim that if $y' > 0$, then there is a curve $\gamma' \subset U$ such that

$$(\gamma' \cap \{z : \operatorname{Re}(z) \geq \xi\}) \subset \{z : 0 < \operatorname{Im}(z) < y'\}.$$

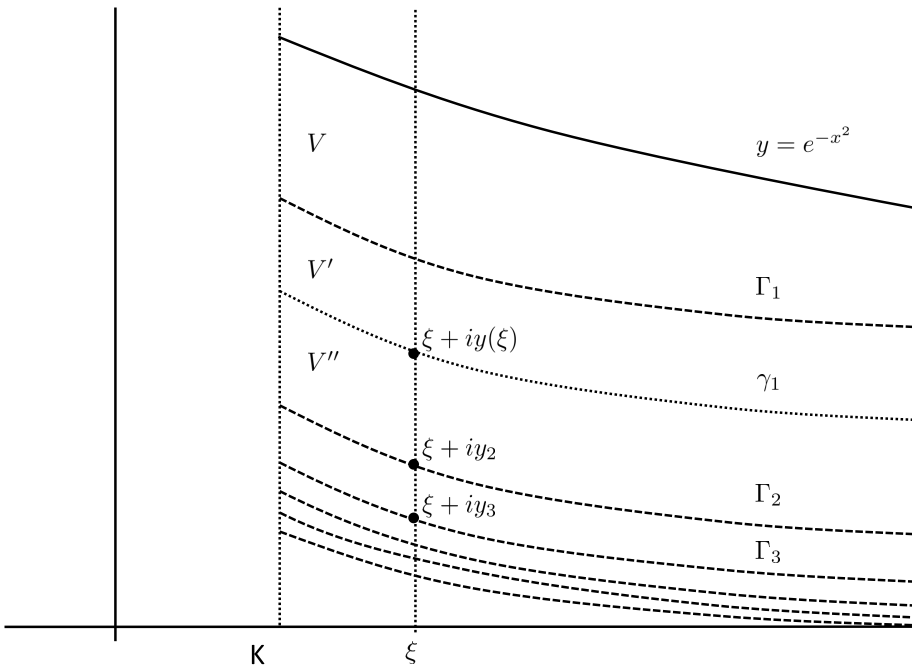

Fig. 2. Some of the preimages of Γ_0, shown dashed, constructed in this proof.

We construct sequences of disjoint curves $(\Gamma_n)_{n\geq 2}$ and positive real numbers $(y_n)_{n\geq 2}$ such that $\Gamma_n \subset U \cap V''$ is an unbounded curve which has a finite endpoint on the line $\{z : \operatorname{Re}(z) = K\}$, for $n \geq 2$; see Figure 2.

We first define Γ_2 and y_2. It follows by (5.11), (5.13) and (5.14) that there exists $0 < y_2 < y(\xi)$ such that $g_\alpha(\xi + iy_2) \in \Gamma_1 \subset U$. Let Γ_2 be the intersection of V'' with the component of $g_\alpha^{-1}(\Gamma_1)$ containing $\xi + iy_2$. By the comments above, Γ_2 cannot lie in a preimage component of U distinct from U, and so must lie in U. Also Γ_2 is unbounded and has a finite endpoint on the line $\{z : \operatorname{Re}(z) = K\}$ by (5.13) and since it cannot intersect with γ_1 or with the real axis.

Suppose next that, inductively, we have defined Γ_k and y_k, for some $k \geq 2$. By (5.13) and (5.14), there exist $0 < y_{k+1} < y_k$ such that $g_\alpha(\xi + iy_{k+1}) \in \Gamma_k \subset U$. Let Γ_{k+1} be the intersection of V'' with the component of $g_\alpha^{-1}(\Gamma_k)$ containing $\xi + iy_{k+1}$. We have that $\Gamma_{k+1} \subset U$ for the same reasons that applied to Γ_2. Finally Γ_{k+1} is unbounded and has a finite endpoint on the line $\{z : \text{Re}(z) = K\}$ by (5.13) and since it cannot intersect with Γ_k or with the real axis. This completes the construction of the sequences.

Suppose that $n \geq 2$. It follows by (5.10) and (5.13) that

$$y_n = \max\{\text{Im}(z) : z \in \Gamma_n \text{ and } \text{Re}(z) \geq \xi\}.$$

It also follows by (5.13) and (5.14) that $y_{n+1} < y_n/8$, for $n \in \mathbb{N}$. This completes the proof of the claim following equation (5.14).

Let W be the component of $\mathbb{C}\backslash U$ containing $[q_\alpha, \infty)$, and let $W' = W\backslash[\xi, \infty)$. We claim that W' is bounded. To prove this we let Γ_1' be the curve in U formed by the union of Γ_1, the complex conjugate of Γ_1 (recall (5.4)), and the part of the line $\{z : \text{Re}(z) = K\}$ joining the finite endpoints of these curves. Let S be the bounded component of

$$\mathbb{C}\backslash(\Gamma_1' \cup \{z : \text{Re}(z) = \xi\}).$$

We claim that, in fact, $W' \subset S$. For, suppose that $z \in W' \cap (\mathbb{C}\backslash S)$. If z lies in the component of $\mathbb{C}\backslash\Gamma_1'$ which does not contain $[q_\alpha, \infty)$, then $\Gamma_1' \subset U$ separates z from $[q_\alpha, \infty)$. On the other hand, if z lies in the component of $\mathbb{C}\backslash\Gamma_1'$ which does contain $[q_\alpha, \infty)$, then $\text{Re}(z) \geq \xi$ and $\text{Im}(z) \neq 0$. Recalling the sequence of disjoint curves $(\Gamma_n)_{n\geq 2}$ constructed earlier, we see that z can be separated from $[q_\alpha, \infty)$ by the curve in U formed by the union of Γ_n, for some $n \in \mathbb{N}$, the complex conjugate of Γ_n, and the part of the line $\{z : \text{Re}(z) = K\}$ joining the finite endpoints of these curves. Hence W' is bounded as claimed.

Let T be the component of $A(g_\alpha)$ containing (q_α, ∞). Suppose, contrary to (5.3), that $T \neq (q_\alpha, \infty)$. Then, since $q_\alpha \notin A(g_\alpha)$, there exists $\zeta \in T$ such that $\text{Im}(\zeta) \neq 0$. We have that $\zeta \in W'$ as otherwise U separates ζ from (q_α, ∞).

It follows from (1.1) that there exist $\ell \in \mathbb{Z}$ and $R > 0$ such that $M^n(R, g_\alpha) \to \infty$ as $n \to \infty$ and $\zeta \in A_R^\ell(g_\alpha)$, where

$$A_R^\ell(g_\alpha) = \{z : |g_\alpha^n(z)| \geq M^{n+\ell}(R, g_\alpha), \text{ for } n \in \mathbb{N},\ n + \ell \in \mathbb{N}\} \subset A(g_\alpha). \tag{5.15}$$

Let T' be the component of $A_R^\ell(g_\alpha)$ containing ζ. We have [21, Theorem 1.1] that T' is closed and unbounded. Since $T' \cap U = \emptyset$ and W' is bounded, T' contains $[\xi, \infty)$. We deduce also that $g_\alpha(T') \subset T'$.

Let $\xi' > q_\alpha$ be the smallest value such that $[\xi', \infty) \subset T'$. Since W' is bounded, by (5.15) there exists $N \in \mathbb{N}$ such that

$$g_\alpha^N(T') \cap W' = \emptyset.$$

Since T' is connected and $g_\alpha^N(T') \subset T'$, we deduce that $g_\alpha^N(T') \subset [\xi, \infty)$. Hence T' contains a curve T'' such that $\{\zeta\} \cup [\xi', \infty) \subset T''$.

Choose ξ'' such that $q_\alpha < \xi'' < \xi'$. Set $\widehat{T} = T'' \cup [\xi'', \xi')$. We note that there exists $N' \in \mathbb{N}$ such that $g_\alpha^{N'}(\widehat{T}) \subset [\xi, \infty)$. Hence there is no neighbourhood of ξ' in which $g_\alpha^{N'}$ is a homeomorphism. This is a contradiction, since $g_\alpha'(x) > 0$, for $x > 0$. This contradiction completes our proof of (5.3). Hence, as already observed, (q_α, ∞) contains uncountably many singleton components of $A''(g_\alpha)$.

Finally we show that $A''(g_\alpha)$ has at least one unbounded component. We recall that

g_α has no multiply connected Fatou component. Hence we may suppose that $\zeta \in A''(g_\alpha)$ is the point constructed in the proof of Theorem 2, with $f = g_\alpha$. We note that since $|g_\alpha(z)| < 1$, for $z \in \Gamma_0$, we may assume that the sequence $(m_k)_{k\in\mathbb{N}}$, also constructed in the proof of Theorem 2, satisfies $m_k = 1$, for $k \in \mathbb{N}$.

Let Q be the component of $A''(g_\alpha)$ containing ζ, and suppose that Q is bounded. Let Q' be the component of $A(g_\alpha)$ containing ζ. Since [21, Theorem 1.1] all components of $A(g_\alpha)$ are unbounded, $Q' \neq Q$. In particular Q' contains a point $\zeta' \in A'(g_\alpha)$.

Since $\zeta' \in A'(g_\alpha)$, there exists $N \in \mathbb{N}$ such that $g_\alpha^N(\zeta') \in (q_\alpha, \infty)$. Since (q_α, ∞) is a component of $A(g_\alpha)$, we obtain that $g_\alpha^N(Q') \subset (q_\alpha, \infty)$. In particular we have that $g_\alpha^N(\zeta) \in (q_\alpha, \infty)$.

We deduce, by (5.2) with $x = g_\alpha^N(\zeta)$, that there exists $N' \in \mathbb{N}$ such that

$$g_\alpha(g_\alpha^n(\zeta)) \geq \frac{1}{e^2} M(g_\alpha^n(\zeta), g_\alpha), \quad \text{for } n \geq N'. \tag{5.16}$$

This is a contradiction to (3.11), and so Q is unbounded as required. □

EXAMPLE 6. We define a family of transcendental entire functions by

$$h_\alpha(z) = \alpha e^z, \quad \text{for } \alpha \in (0, e^{-1}).$$

This family is contained in the *exponential family*, the dynamics of which have been widely studied. It is well-known that h_α has an unbounded simply connected Fatou component, which contains the imaginary axis and an attracting fixed point, and also a repelling fixed point $q > 1$.

As in Example 1, we see that $(q, \infty) \subset A'(h_\alpha)$. The techniques of the proof of Lemma 5·2 may be used to show that (q, ∞) is a component of $A(h_\alpha)$. We omit the details.

Acknowledgements. The author is grateful to Gwyneth Stallard and Phil Rippon for all their help with this paper, to John Osborne for his drawing his attention to [14], and to the referee for a number of helpful suggestions.

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